

An Example of Quantum Anomaly in the Physics of Ultra-Cold Gases

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In this article, we propose an experimental scheme for observation of a quantum anomaly—quantum-mechanical symmetry breaking—in a two-dimensional harmonically trapped Bose gas. The anomaly manifests itself in a shift of the monopole excitation frequency away from the value dictated by the Pitaevskii-Rosch dynamical symmetry [L. P. Pitaevskii and A. Rosch, Phys. Rev. A, **55**, R853 (1997)]. While the corresponding classical Gross-Pitaevskii equation and the derived from it hydrodynamic equations do exhibit this symmetry, it is—as we show in our paper—violated under quantization. The resulting frequency shift is of the order of 1% of the carrier, well in reach for modern experimental techniques. We propose using the dipole oscillations as a frequency gauge.

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Introduction.— The current decade is marked by the emerging links between ultracold-atom physics on one hand and cosmology and high-energy physics on another. Examples include kinetics of black holes [1], electron-positron pair production [2], *Zitterbewegung* [3], and the string theory limits posed on viscosity [4, 5]. However, quantum anomalies are usually considered to be a purely quantum-field-theoretical phenomenon [6, 7]. In this article, we suggest a scheme for observing a quantum anomaly in ultracold two-dimensional harmonically trapped Bose gas.

Quantum anomaly—otherwise known as the quantum mechanical symmetry breaking—consists of three ingredients. First ingredient is an exact symmetry in the classical version of the theory in question. Second is a divergence that appears in the straightforward quantum version of the theory. Third ingredient is a weak violation of the original symmetry that emerges in a regularized version of the quantum theory.

The two-dimensional δ -potential has been long recognized as an example of quantum anomaly in elementary quantum mechanics [8, 9]. The classical symmetry of the δ_2 -potential originates from the following property: under the scaling transformation $\mathbf{r} \rightarrow \lambda \mathbf{r}$, the potential transforms in exactly the same way as the kinetic energy does. A consequence of this property is the absence of any length scale in the corresponding dynamical problem, both before and after a straightforward quantization. Next, an analysis of the scattering properties of the δ_2 -potential [10, 11] shows a divergence in an all Born orders of the scattering amplitude starting from the second. Finally, the subsequent regularization [10, 11] leads to the appearance of the new length scale a_{2D} . The original symmetry becomes broken, and a quantum anomaly emerges.

Similar anomaly in the case of the $1/r^2$ potential was discussed in Ref. [12], along with various regularization methods [13, 14].

In Ref. [15], Pitaevskii and Rosch predicted a dynamical symmetry that appears in the classical field theory (Gross-Pitaevskii equation) of the δ_2 -interacting two-dimensional harmonically trapped Bose gas. This symmetry is a direct consequence of the scaling symmetry of the δ_2 -potential, described above. The consequences of this symmetry are (a) absence of the amplitude dependence of the main frequency (isochronicity) and (b) absence of higher overtones in the time dependence (monochromaticity) of the monopole oscillations of the moment of inertia. Both properties were demonstrated experimentally [16], along with an anomalously slow damping, for the case of a very elongated Bose-Einstein condensate; its Gross-Pitaevskii equation coincides with the one for a two-dimensional condensate.

In this article we address the question of *whether the Pitaevskii-Rosch symmetry survives quantization*.

In the fully quantized unitary gas, the analogous symmetry has been shown to remain unbroken [17]. The quantum correction to the frequency of the monopole excitation in an elongated condensate [16] was computed in Ref. [18]

The zero-temperature equation of state of the two-dimensional Bose gas.— The low-density zero-temperature quantum field theory (QFT) expression for the chemical potential of the two-dimensional Bose gas is well known [19, 20]: it reads

$$\mu(n) \stackrel{\text{QFT}}{=} \frac{4\pi\hbar^2}{m} \chi(\pi e^{2\gamma+1} n a_{2D}^2) n, \quad (1)$$

where a_{2D} is the two-dimensional scattering length, $\gamma = 0.5772\dots$ is the Euler's constant, n is the two-dimensional density, and $\chi(z) = \frac{1}{-W_{-1}(-z)}$. For a given two-dimensional two-body interaction potential $V(\mathbf{r})$, its scattering length a_{2D} is defined as the radius of a hard disk whose zero-energy s -wave scattering amplitude equals the one for the potential V . Here, $W_{-1}(z)$ is the minus-first branch of the Lambert's W -function

[21]. For the small values of the gas parameter na_{2D}^2 , the factor χ is a logarithmically slow function of the density: $\chi(z) \xrightarrow{z \rightarrow 0} 1/\ln(1/z) + \mathcal{O}(\ln(\ln(1/z))/\ln(1/z)^2)$. The expression (1) is an inverse of a more conventional formula $n(\mu) = (m\mu/4\pi\hbar^2) \ln(4\hbar^2/e^{2\gamma+1}m\mu a_{2D}^2)$, shown to be the leading term in an expansion in powers of $1/\ln(4\hbar^2/e^{2\gamma+1}m\mu a_{2D}^2)$ [22].

In the case of three-dimensional short-range-interacting atoms tightly confined to a two-dimensional plane by a one-dimensional harmonic potential, the two-dimensional scattering length can be expressed through the three-dimensional scattering length a_{3D} and the confinement size $\tilde{a}_z = \sqrt{2\hbar/(m\omega_z)}$ as $a_{2D} = C_{2D}\tilde{a}_z \exp[-(\sqrt{\pi}/2)(\tilde{a}_z/a_{3D})]$, where $C_{2D} = 1.4795\dots$ [23, 24]. The chemical potential becomes $\mu(n) \stackrel{\text{QFT}}{=} \bar{g}_{2D} n (\chi(\sigma(n)e^{-\frac{1}{\epsilon}})/\epsilon)$ where $\bar{g}_{2D} = 4\pi\epsilon\hbar^2/m$ is the “bare” two-dimensional coupling constant, $\epsilon = a_{3D}/\sqrt{\pi}\tilde{a}_z$ is the *small parameter governing the proximity to the classical limit*, and $\sigma(n) = \pi e^{2\gamma+1}(C_{2D})^2 n \tilde{a}_z^2$. In the limit $\epsilon \ll \min(1, 1/|\ln(\sigma(n))|)$, the factor χ approaches the density-independent constant ϵ , and the chemical potential converges to the prediction of the classical field theory (CFT), otherwise known as the Gross-Pitaevskii equation; there, the chemical potential reads

$$\mu(n) \stackrel{\text{CFT}}{=} \bar{g}_{2D} n \quad . \quad (2)$$

Hydrodynamic equations.– The zero-temperature hydrodynamic (HD) equations for a two-dimensional harmonically trapped Bose gas read

$$\frac{\partial}{\partial t} n + \nabla_{\mathbf{r}}(n\mathbf{v}) = 0 \quad (3)$$

$$\frac{\partial}{\partial t} \mathbf{v} + (\mathbf{v} \cdot \nabla_{\mathbf{r}})\mathbf{v} = -(1/m)\nabla_{\mathbf{r}}[\mu(n) + V_{\text{HO}}(\mathbf{r})], \quad (4)$$

where m is the atomic mass, $n = n(\mathbf{r}, t)$ is the atomic density, $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ is the atomic velocity, $V_{\text{HO}}(\mathbf{r}) = m\omega^2 r^2/2$ is the trapping potential energy per particle, $\mu(n)$ is the chemical potential, and $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y$. Assuming that the gas is a Bose condensate with no vortices, the velocity field can be assumed to be irrotational: $\mathbf{v} = \hbar\nabla\Phi/m$, where $\Phi(\mathbf{r})$ is the potential of the velocity field. Under this assumption, the HD equations (3,4) can be rewritten in a Hamiltonian form, $\frac{\partial}{\partial t} n(\mathbf{r}, t) = \frac{i}{\hbar}[H, n(\mathbf{r}, t)]_{\text{HD}}$, $\frac{\partial}{\partial t} \Phi(\mathbf{r}, t) = \frac{i}{\hbar}[H, \Phi(\mathbf{r}, t)]_{\text{HD}}$. The Hamiltonian is represented by a sum of two parts, the first being in turn a sum of the kinetic and interaction energies and the second being the trapping energy: $H = H_0 + H_{\text{HO}}$. Here $H_0 = \int d^2\mathbf{r} \left\{ \frac{\hbar^2}{2m} n(\nabla\Phi)^2 + \varepsilon(n) \right\}$, $H_{\text{HO}} = \int d^2\mathbf{r} V_{\text{HO}}(\mathbf{r})$, the “hydrodynamic commutator” $[\cdot, \cdot]_{\text{HD}}$ is given by the Poisson brackets with respect to

the (n, Φ) canonical pair,

$$[A(n(\mathbf{r}, t), \Phi(\mathbf{r}, t)), B(n(\mathbf{r}', t), \Phi(\mathbf{r}', t))]_{\text{HD}} = -\frac{i}{\hbar} \times \left[\left(\frac{\partial}{\partial n} A(n(\mathbf{r}, t), \Phi(\mathbf{r}, t)) \right) \left(\frac{\partial}{\partial \Phi} B(n(\mathbf{r}, t), \Phi(\mathbf{r}, t)) \right) - \left(\frac{\partial}{\partial \Phi} A(n(\mathbf{r}, t), \Phi(\mathbf{r}, t)) \right) \left(\frac{\partial}{\partial n} B(n(\mathbf{r}, t), \Phi(\mathbf{r}, t)) \right) \right] \times \delta_2(\mathbf{r} - \mathbf{r}') \quad ,$$

and $\varepsilon(n) = \int_0^n dn' \mu(n')$ is the microscopic energy density.

The Pitaevskii-Rosch symmetry and the quantum anomaly at the HD level.– Introducing the generator of the scaling transformations, $Q = \frac{1}{\hbar} \int d^2\mathbf{r} n(\mathbf{r} \cdot \nabla\Phi)$ (see Ref. [15] for example), one obtains the following set of commutation relations:

$$[Q, H_0]_{\text{HD}} \stackrel{\text{QFT}}{=} 2iH_0 + ia_{2D} \frac{\partial}{\partial a_{2D}} H_0 \quad , \quad (5)$$

$[Q, H_{\text{HO}}]_{\text{HD}} \stackrel{\text{QFT}}{=} -2iH_{\text{HO}}$, $[H_{\text{HO}}, H_0]_{\text{HD}} \stackrel{\text{QFT}}{=} i\omega^2 Q$. Notice that the classical field theory chemical potential (2) does not depend on the two-dimensional scattering length a_{2D} . Then, the commutator (5) becomes

$$[Q, H_0]_{\text{HD}} \stackrel{\text{CFT}}{=} 2iH_0 \quad . \quad (6)$$

In this case, the observables H_0 , H_{HO} , and Q form a closed three-dimensional algebra, identical to the one discovered by Pitaevskii and Rosch [15] at the Gross-Pitaevskii equation level.

However, the more accurate quantum field theory prediction for the chemical potential (1) depends explicitly on a_{2D} . According to the Eqn. 5, the classical commutation relation (6) becomes corrected by $ia_{2D} \frac{\partial}{\partial a_{2D}} H_0$. Since the correction term is not generally expected to be a function of three original members of the algebra, the algebra opens; such an opening constitutes a *quantum anomaly*.

Castin-Dum-Kagan-Surkov-Shlyapnikov equations.– If the factor χ in the chemical potential expression (1) was a constant, the HD equations (3,4) could be solved via the Castin-Dum-Kagan-Surkov-Shlyapnikov (CDKSS) scaling ansatz [25, 26]. Note however, that χ is a very slow function of the density, and thus the scaling ansatz should approximately hold. Consider the following ansatz: $n(\mathbf{r}, t) = \frac{1}{\lambda(t)^2} n_0 [1 - (r/(\lambda(t)R))^2]$, $\mathbf{v}(\mathbf{r}, t) = \mathbf{e}_r r \frac{\dot{\lambda}(t)}{\lambda(t)}$, where $\tilde{R}_{TF} = \sqrt{2g_{2D}(n_0)n_0/m\omega^2}$ is the steady-state HD curvature of the density distribution (that also corresponds to the Thomas-Fermi radius of a gas with a density-independent coupling constant fixed to $g_{2D}(n_0)$), $g_{2D}(n) \equiv (\partial\mu/\partial n)$ is an effective density-dependent coupling constant, n_0 is the steady-state peak density; the scaling parameter λ obeys the CDKSS equation

$$\ddot{\lambda} = \frac{\omega^2 u(\lambda)}{\lambda^3} - \omega^2 \lambda \quad (7)$$

with $u(\lambda) = g_{2D}(n_0/\lambda^2)/g_{2D}(n_0)$. It can be shown that the above ansatz solves the HD equations (3,4) almost everywhere, with the exception of an exponentially narrow ring close to the edge of the cloud.

For example, let us define the thickness $\Delta R(\delta)$ of a ring of positions \mathbf{r} such that the effective coupling $g_{2D}(n(r))$ differs from that in the center by a relative correction δ or more, but the density is still greater than zero: $|g_{2D}(n(r))/g_{2D}(n_0) - 1| > \delta$ and $r < R_{TF}$. (Here, R_{TF} is the “true” Thomas-Fermi radius defined by $V_{HO}(R_{TF}) = \mu(n_0)$.) Then one can show that in the limit $\epsilon \rightarrow 0$, $\sigma(n) = \text{const}$, the leading behavior of the thickness $\Delta R(\delta)$ is $\Delta R(\delta) = (1/2) \exp(-\delta/\epsilon) R_{TF}$. Likewise, one can show that the curvature radius \tilde{R}_{TF} is exponentially close to the Thomas-Fermi radius of the cloud: $\tilde{R}_{TF} \approx R_{TF}$.

Anomalous frequency shift of the monopole frequency.—Linearization of the equation (7) for small excitation amplitudes readily gives the frequency of small oscillations, $\Omega = \Omega_0(1 + \delta(\text{AR}_\lambda = 0))$, where [27]

$$\delta(\text{AR}_\lambda = 0) = \frac{1}{4} \frac{\chi(\sigma(n_0)e^{-\frac{1}{\epsilon}})}{(1 - \chi(\sigma(n_0)e^{-\frac{1}{\epsilon}}))^2} . \quad (8)$$

The deviation of the monopole frequency Ω from the classical field theory prediction $\Omega_0 \equiv \Omega|_{\epsilon \ll \min(1, 1/|\ln(\sigma(n))|)} = 2\omega$ is a manifestation of the quantum anomaly.

Here and below, $\text{AR}_\lambda = (\lambda_{\max} - \lambda_{\min})/(\lambda_{\max} + \lambda_{\min}) = (\lambda_{\max}^2 - 1)/(\lambda_{\max}^2 + 1)$ is the aspect ratio for the monopole oscillations.

For $\epsilon \ll \min\left(1, \frac{1}{|\ln(\sigma(n_0))|}\right)$, the relative anomalous correction converges to

$$\delta(\text{AR}_\lambda = 0) \approx \frac{1}{4\sqrt{\pi}} \frac{a_{3D}}{\tilde{a}_z} . \quad (9)$$

In particular, this estimate shows that the effect of the anomaly can be enhanced using a Feshbach resonance.

Furthermore, a numerical analysis shows that under quantization, the monopole frequency becomes amplitude-dependent. Fig. 1 shows the corresponding prediction of the CDKSS equation (7) compared to both the results of the full HD treatment (3,4) and the limiting values (9).

A possible experimental scheme for detecting the anomalous frequency shift.—Consider the following excitation scheme. Initially, the cloud is prepared in the ground state of a frequency ω_{init} trap at a position $x = 0, y = 0$. At $t = 0$, the trap is simultaneously relaxed to a lower frequency ω and shifted to a new position $x = x_0, y = 0$. This initial condition will induce a superposition of a monopole oscillation that starts from the lowest in cloud size turning point and a dipole oscillation—whose frequency ω does not depend on either interaction strength or the oscillation

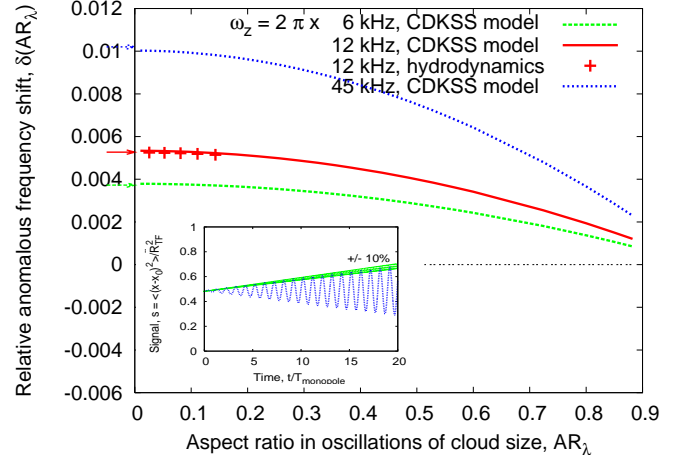


FIG. 1: (color online). Relative anomalous shift of the monopole frequency, $\delta(\text{AR}_\lambda) \equiv (\Omega - \Omega_0)/\Omega_0$ as a function of the amplitude of the excitation, for different values of the transverse confinement frequency ω_z . Solid lines: prediction of the modified Castin-Dum-Kagan-Surkov-Shlyapnikov (CDKSS) model (7). Crosses: prediction of the HD equations (3,4). Arrows: the $a_{3D} \ll \tilde{a}_z$ limit (9) of the CDKSS model prediction (8) for the zero-amplitude shift. The rest of the parameters corresponds to $N = 10^3$ rubidium 87 atoms ($a_{3D} = 5.2$ nm) confined longitudinally in a frequency ω harmonic trap, where $\omega = 2\pi \times 41$ Hz for the $\omega_z = 2\pi \times 45$ kHz data and $\omega = 2\pi \times 40$ Hz for the rest of the data. The steady-state peak density n_0 entering the CDKSS model was derived from the number of atoms N using the Euler equation (4). An possible experimental setup corresponding to the 45 kHz curve is described in detail in the concluding section of this article. Insert shows the monopole vs. dipole beat signal for the optimal choice of the initial trap shift (see main text). Parameters correspond to the 45 kHz curve taken at the aspect ratio $\text{AR}_\lambda = 0.486$. We also show the initial linear part of the slow envelope, $(\lambda_{\max}^2/6 + ((\lambda_{\max}^4 - 1)/(12\lambda_{\max}^2)))\delta(\text{AR}_\lambda = 0.486)\Omega_0 t$ plotted with a $\pm 10\%$ uncertainty in the value of the anomalous correction $\delta(\text{AR}_\lambda) = 0.00760$. The CDKSS model was used for calculations.

amplitude—that starts from the left(right) turning point for $x_0 > 0(x_0 < 0)$.

What we suggest is to measure the spatial mean of the square of the horizontal displacement with respect to the new trap center, $s(t) = \int dx dy (x - x_0)^2 n(x, y, t)$. After a lengthy but straightforward calculation it can be shown that this observable will evolve in time as $s(t)/\tilde{R}_{TF}^2 = P + A \cos[\Omega_0 t] + B \cos[\Omega_0(1 + \delta(\text{AR}_\lambda))t + \phi]$, where $P = (\lambda_{\max}^4 + 1)/12\lambda_{\max}^2 + \eta^2/2$, $A = \eta^2/2$, $B = (\lambda_{\max}^4 - 1)/12\lambda_{\max}^2$, $\phi = \pi$, $\eta = x_0/\tilde{R}_{TF}$, and \tilde{R}_{TF} is the effective Thomas-Fermi radius for the final, frequency ω trap (see a definition above). Choosing $\eta = \sqrt{\frac{\lambda_{\max}^4 - 1}{6\lambda_{\max}^2}}$ leads to $A = B$. It is easy to see that in this case, at $t = 0$ the beats between the dipole and monopole oscillations have a node, with the first crest reached in $1/(2\delta(\text{AR}_\lambda))$ monopole periods. In terms of the initial trap, the optimal shift of the trap can

be expressed as $x_0 = \frac{1}{\sqrt{6}} \sqrt{\left(\frac{\omega_{\text{init.}}}{\omega}\right)^2 - 1} \tilde{R}_{TF, \text{init.}}$, where $\tilde{R}_{TF, \text{init.}} = \sqrt{\omega/\omega_{\text{init.}}} \tilde{R}_{TF}$ is the Thomas-Fermi radius for the *initial*, frequency $\omega_{\text{init.}}$ trap. Note as well that the right turning point for the oscillations of the scaling parameter can be expressed as $\lambda_{\text{max}} = \sqrt{\omega_{\text{init.}}/\omega}$. In our excitation scheme, the monopole oscillations start from the left turning point that in turn corresponds to the lowest cloud size. Insert of Fig. 1 presents an example of the projected beat signal for a typical set of parameters.

The present in any realistic trap anharmonicity and anisotropy may complicate the matters. For the anharmonicity, one can show that for a quartic correction of a form $m\omega^2 r^2/2 \rightarrow (m\omega^2 r^2/2)(1 + \alpha r^2/2a_0^2)$ ($a_0 = \sqrt{\hbar/m\omega}$ being the size of the ground state), the relative (to the monopole frequency) shift of the monopole frequency is $\delta_{\text{anharm.}} = \frac{12}{11} \alpha \sqrt{N} \epsilon$, to the leading order in both α and the amplitude. The doubled dipole frequency—that serves as a reference—is unshifted in that order. On the other hand, the anisotropy leads to a splitting of the dipole frequency with the monopole frequency situated in between the two resulting frequencies. If one chooses the doubled X-dipole frequency $2\omega_x$ as a reference, the relative to this frequency monopole shift will be given, to the lowest order in both anisotropy $\omega_x - \omega_y$ and the amplitude, by $\delta_{\text{anisotr.}} = (\omega_x - \omega_y)/2\omega_{x,y}$, where $\omega_{x,y}$ is any of the two frequencies. Both the anharmonicity ($\delta_{\text{anharm.}}$) and anisotropy ($\delta_{\text{anisotr.}}$) shifts must be kept below 1% to allow for observation of the quantum anomaly.

Vortex-antivortex pair creation has been suggested as the dominant mechanism for damping of the two-dimensional monopole oscillations [2], all the conventional channels being suppressed due to the Pitaevskii-Rosch symmetry. Here, the amplitude of the oscillations decays as $\lambda_{\text{max}}(t) = \lambda_{\text{max}}(0)/(1 + t/\tau)^{1/10}$, where $\tau = \frac{737280}{127} \left(\frac{\omega_{\text{init.}}}{\omega}\right)^5 \left(\frac{\lambda_{\text{min}}(0)}{\lambda_{\text{max}}(0)}\right)^{10} \frac{\ln(4m\bar{g}_{2D}N/\sqrt{\pi}\hbar^2)}{(m\bar{g}_{2D}/\hbar^2)^{9/4}\sqrt{N}}$. Here, it is assumed that the amplitude of oscillations is large (i.e. $\lambda_{\text{max}} \gg 1$) and the gas is dilute (i.e. $a_{3D} \ll \tilde{a}_z$).

Let us now propose a concrete example of a trap suited for an observation of the quantum anomaly. The numbers proposed are those of rubidium 87 in its $5S_{1/2}$, $F = 2$, $m_F = 2$ ground state. A blue detuned standing wave (wavelength 532 nm, power P , waist w_0) crosses, at a right angle, the symmetry axis z of a quadrupolar magnetic field; the magnetic field gradients are given by b' in the xy plane and $-2b'$ along z . This results in a series of pan-cake traps: there, the magnetic field is responsible for the weak horizontal trapping, while the standing wave gives a strong vertical confinement. Atoms could be loaded in one of the nodes of the standing wave, at a distance $z = d$ from the trap center; there the magnetic field is bounded from below by $B_0 = 2b'd$. By construction, this trap should be isotropic in the horizontal plane. Furthermore, the residual anisotropy defects

could be further compensated by adding small magnetic gradients in the horizontal plane. For $b' = 167 \text{ G/cm}$, $d = 400 \mu\text{m}$, $P = 4 \text{ W}$, and $w_0 = 500 \mu\text{m}$, the oscillation frequencies are $41 \text{ Hz} \times 41 \text{ Hz} \times 45 \text{ kHz}$, and the magnetic field minimum $B_0 = 13.4 \text{ G}$ well prevents any Majorana losses. The photon scattering rate is as low as $2.6 \times 10^{-3} \text{ s}^{-1}$, corresponding to 30,000 monopole mode oscillations. For $N = 10^4$ atoms (ten times the number used in Fig. 1), the anharmonic parameter is bounded from above by $\alpha = -2.1 \times 10^{-6}$. The predicted Thomas-Fermi radius of $15 \mu\text{m}$ is much smaller than both d and w_0 ; accordingly, the anharmonic shift has a negligibly small value of $\delta_{\text{anharm.}} = 5 \times 10^{-5}$. The chemical potential of $\mu/\hbar = 1.7 \text{ kHz}$ is well below the transverse frequency; this ensures that the trap is well in the two-dimensional regime. Finally, taking oscillation aspect ratio of $\text{AR}_\lambda = 0.486$ as an example, one obtains the damping time τ which is 1576 times longer than the period of the monopole oscillations. Overall, these figures bring our proposal within reach of modern experimental technology.

Summary and outlook.— In this article, we propose an experimental scheme for observation of a quantum anomaly in a two-dimensional harmonically trapped Bose gas. The effect consists of a shift of the monopole excitation frequency away from the value dictated by the Pitaevskii-Rosch dynamical symmetry [15]. The shift we predict is only of the order of 1% of the base frequency. To detect it, we propose using the dipole oscillations as a reference frequency.

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